SOLUTION OF NONSTATIONARY HEAT AND MASS CONDUCTIVITY PROBLEMS BY USING THE IMAGINARY-FREQUENCY CHARACTERISTICS

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A method for the approximate solution of the internal problem of nonstationary heat and mass conductivity is elucidated which is based on a rational fraction mode of approximating transcendental functions. Computational formulas are presented for one-dimensional plates, cylinders, and spheres.

The extensive application of the heat-conduction equations to model the most diverse technological, thermal, electrical, and other processes predetermines the need to develop engineering methods for their solution that would assure adequate accuracy for practical purposes.

Inverse problems that are of great applied value can also be solved efficiently by approximate methods.

The main difficulty in performing investigations is in solving the internal problem [1] that sets up a relationship between the functions inside the body volume U(r, t) and on the surface $U_n(R, t)$ since the process is described by partial differential equations with boundary conditions of the first kind. The dependence of $U_n(R, t)$ on the source of perturbations $U_c(t)$ (external problem) is modeled in simple form as a rule. Consequently, it is necessary to have primarily an approximate solution of the internal problem.

If the Laplace transform is used in a linear formulation, then its exact solution has the form

$$L\,[U(\mathbf{r},\ t)-U_0(\mathbf{r},\ 0)]=\bar{\Phi}(\mathbf{r},\ S)\cdot L\,[U_n(\mathbf{R},\ t)-U_{n0}(\mathbf{R},\ 0)], \tag{1}$$

where U_0, U_{n0} are initial conditions and $\bar{\Phi}(r, S)$ is the transfer function.

It is shown in [2] that utilization of the concepts and properties of transfer functions substantially extends the possibilities of heat- and mass-transfer analysis.

As is known, $\bar{\Phi}(r, S)$ is found considerably more simply than the original $\Phi(r, t)$, the pulse transfer function, which indeed yields the solution of the problem in the form of the integral

$$U(\mathbf{r},\ t)=U_0(\mathbf{r},\ 0)+\int_0^t[U_n(\mathbf{R},\ x)-U_{n0}(\mathbf{R},\ 0)]\,\Phi(\mathbf{r},\ t-x)\,dx. \tag{2}$$

The functions $\bar{\Phi}(r, S)$ are transcendental, and have an infinite number of poles on the negative semiaxis of the complex variable S, which yields an infinite series with exponential terms upon going over to $\Phi(r, t)$. The construction and analysis of this series indeed comprises the fundamental complexity of the exact solution. The search for an approximate solution is therefore to approximate the exact transfer function by a simpler expression $\bar{\Phi}_{ap}(r, S)$.

Such methods have been developed in automatic regulation theory. In particular, an approximation method based on analyzing the imaginary-frequency characteristics (IFC) of the transfer function has been proposed in [3]. The IFC is the curve obtained in slitting $\bar{\Phi}(r, S)$ along the real semiaxis of the variable S. It is proved in [3] that the error in $\Phi_{ap}(r, t)$ does not exceed the error in approximating the "exact" function $\bar{\Phi}(r, S)$ by its approximate expression $\bar{\Phi}_{ap}(r, S = \xi)$ (where $0 \leq \xi \leq \infty$ are real numbers), if $\Phi(r, t)$ contains no high-frequency components. As a rule, the functions $\Phi(r, t)$ describing the heat and mass-transfer processes do not generally contain harmonic terms [4] and, consequently, they satisfy the

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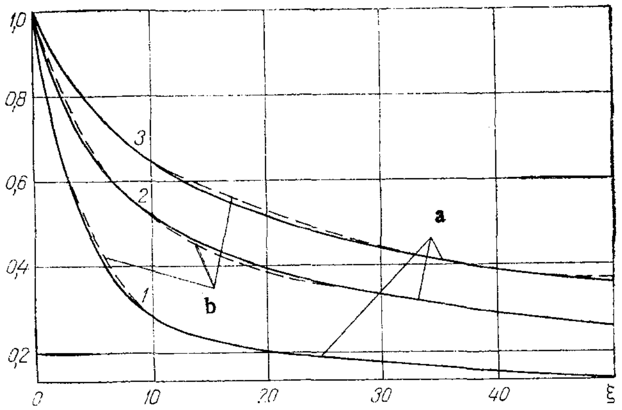

Fig. 1. Imaginary-frequency characteristics for the mean temperature of a plate (1), cylinder (2), sphere (3), exactly (a) and approximately (b).

TABLE 1. Values of the Coefficients

Body	D	A	B	C
Plate	1	0,43	0,1	0,01
Cylinder	1	0,1836	0,0856	0,00184
Sphere	1	0,1128	0,0461	0,0011

requirement mentioned. The IFC for $\bar{\Phi}(r, S = \xi)$ has a monotonically decreasing aperiodic form and it is expedient to take the approximating function $\bar{\Phi}_{ap}(r, S)$ in the rational fraction form

$$\bar{\Phi}_{ap}(r, S) = D(r)\,\frac{1 + B(r)S}{1 + A(r)S + C(r)S^2}. \tag{3}$$

If B = C = 0 is taken, then we obtain the well-known quasistationary model by which it is impossible to describe satisfactorily both the low- and the high-frequency spectral ranges of the operator $\bar{\Phi}(r, S)$ and consequently, such approximations have significant errors in the case of large transient velocities. The second-order model (3) permits obtaining a solution with 3-5% error in the whole frequency range.

The coefficients A-D are found by sampling from the minimal deviation condition for the curves $\bar{\Phi}(r, \xi)$ and $\bar{\Phi}_{ap}(r, \xi)$, particularly in the domain of strong variations (close to the origin). It is here desirable to utilize the equality of these functions and their first derivatives at $\xi = 0$. It is recommended [3] to use the least-square deviation method to sample the coefficients.

As the original of (3), the pulse-transfer function has the form

$$\Phi_{ap}(r, t) = b_1 \exp(-S_1 t) + b_2 \exp(-S_2 t), \tag{4}$$

where

$$S_{1,2} = \frac{A \mp \sqrt{A^2 - 4C}}{2C};$$

$$b_1 = \frac{D}{C}\,\frac{1 - BS_1}{S_2 - S_1}; \quad b_2 = \frac{D}{C}\,\frac{S_2 B - 1}{S_2 - S_1}.$$

The first term describes the low-frequency domain (regular regimes) and the second the high-frequency domain. It is indubitable that utilization of (4) in (2) in place of the infinite series will substantially simplify the analysis of transients. In investigating nonstationary processes, it is useful to compute the mean function $U_m = \frac{1}{V}\int_V U dV$ over the volume element of V, that governs the change in the thermal or mass fluxes, for which the appropriate integration must be performed in (1). As an illustration, values of the coefficients A-D are given in the table for the functions connecting U_m and U_n for a one-dimensional symmetric

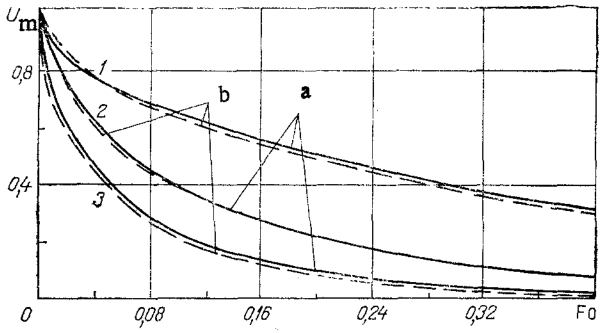

Fig. 2. Change in the mean temperature for a unit step in the surface temperature of a plate (1), cylinder (2), and sphere (3), exactly (a) and approximately (b).

plate $\bar{\Phi}^{pl} = \frac{th(\sqrt{S})}{\sqrt{S}}$, cylinder $\bar{\Phi}^{cyl} = \frac{2I_1(\sqrt{S})}{\sqrt{S}\,I_0(\sqrt{S})}$, and sphere $\bar{\Phi}^{sph} = 3\left[\frac{1}{\sqrt{S}\,th(\sqrt{S})} - \frac{1}{S}\right]$ (here the Fourier criterion was taken as the time).

Exact and approximate IFC for these cases are presented in Fig. 1, and the corresponding transient curves for jump perturbations in Fig. 2. The errors in the approximate solutions do not exceed 2-3%.

The dependence of $U_n(R, t)$ on $U(r, t)$ is easily obtained from (3) in both the operator and time forms

$$U_n(\mathbf{R}, t) = U_{n0}(\mathbf{R}, 0) + a_1 \frac{dU_1(\mathbf{r}, t)}{dt} + a_2 [U(\mathbf{r}, t) - U_0(\mathbf{r}, 0)] + a_3 \int_0^t [U(\mathbf{r}, x) - U_0(\mathbf{r}, 0)] \exp\left[-\frac{(t-x)}{B}\right] dx, \quad (5)$$

where $a_1 = C/DB$, $a_2 = [A - (C/B)]1/DB$; $a_3 = (1 - a_2)1/DB$, which is not satisfied successfully in the exact formulation since there are no originals of $\Phi^{-1}(r, S)$. Relationship (5) permits a sufficiently simple solution of inverse heat- and mass-transfer problems. An essential distinction in the operators solving the direct and inverse problems follows from a comparison of (2) and (5). Exact solutions of inverse problems [1] are represented by infinite series in derivatives of U(r, t), which indeed determines the complexity of their practical application, while the approximate model uses just the first derivative, which can be determined from experimental curves to sufficient accuracy.

For the complete solution of the problem it is necessary to add the appropriate boundary conditions of the second, third, and fourth kinds to the obtained approximate solution of the internal problem in the form (1), (2), (3), which results in a system of equations whose integration is substantially simpler as compared with the originals. Investigations performed in [5, 6] displayed the high efficiency of the method elucidated.

NOTATION

t, time; r, space coordinate vector; S, Laplace transformation variable; R, body surface vector; L, Laplace transform operator.

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